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NORMAL AND TANGENTIAL COMPLIANCE FOR CONFORMING BINDER CONTACT II: VISCO-ELASTIC BINDER

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Abstract—This article extends the previously derived elastic solutions to visco-elastic solutions of normal and tangential compliances for a system comprised of two elastic particles bonded by a thin layer of visco-elastic binder. Rate dependent compliances are derived for both Maxwell and Voigt visco-elastic binders. Similar to the elastic particle-binder system, the time-dependent governing equation of this problem is also a Fredholm integral equation of the second kind. We employ the monotonical property of the *kernel function* to obtain the upper and lower bounds for the rate-dependent compliance relationship. Guided from the upper and lower bound solutions, we derive the best estimated solution based on physically consistent approximations. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

A companion paper (Zhu *et al.*, 1995) presented methods of deriving the compliance relationship for an elastic particle–binder system. In this paper the methods are extended to derive the compliance relationship for a system of two elastic particles bonded together by a thin layer of visco-elastic binder. Binders in granular/particulate materials are usually viscous to a certain degree. For example, binders exhibit strong viscosity in asphaltic concrete and moderate viscosity in cemented sand. This article makes an effort in exploring the characteristics of viscous compliance.

In the past years, many topics on elastic binder/layer contact have been investigated (for example, Goodman and Keer, 1975; Meijers, 1968; Alblas and Kuipers, 1970; Matthewson, 1981; Keer *et al.*, 1991; Dvorkin *et al.*, 1994; Johnson, 1985; Gladwell, 1980, etc.). However, little published work can be found on the viscous binder/layer contact, except a small number of articles in Russian literature (Manzhirov, 1983; Aleksandrov *et al.*, 1989).

In this study, we aim to derive closed-form time-dependent relationships between the contact forces and the relative particle/binder movements in this system. We begin the article with an establishment of governing integral equations that dictate the time-dependent interfacial contact pressure distribution between the elastic particle and the visco-elastic binder. We then derive the closed-form compliance relationships for both Maxwell and Voigt types of visco-elastic binders. We pursue the solutions using the monotonical properties of the kernel function. The derived rate-dependent compliances of the two particle system include the upper bound solution, the lower bound solution and the best estimated solution.

FORMULATION OF THE PROBLEM

In this section, we illustrate the compliance of two particles with a thin layer of visco-elastic binder based on the formulation for the elastic system (Dvorkin *et al.*, 1994). The contact configuration of two particles bonded by a binder is same as that described previously in the elastic system (Zhu *et al.*, 1995). The interfacial boundary between the particles and the binder is axi-symmetric, given by $z = h(r)$:

$$h(r) = h_0\left(1+d\frac{r^2}{a^2}\right) \tag{1}$$

where a is the radius of contact area, h_0 is the thickness of the binder at $r = 0$, and the dimensionless shape parameter d is limited in a range $0 \leqslant d < 1$ to represent the curvature of particle surface varying from flat to spherical.

We denote the constraint moduli E_1 and E_2, the shear moduli G_1 and G_2, Poisson's ratio v_1 and v_2 for the particles and the binder respectively. Here, we consider two types of visco-elastic binders, namely Maxwell and Voigt models. For the Maxwell model, the normal stress–strain relationship in the thin layer of binder is given by

$$\dot{\varepsilon}_2(r,t) = \frac{1}{E_2}\dot{p}(r,t)+\frac{1}{\eta_\varepsilon}p(r,t) \tag{2}$$

or in its integral representation, the relation reads

$$\varepsilon_2(r,t) = \frac{p(r,t)}{E_2}+\frac{1}{\eta_\varepsilon}\int_0^t p(r,\tau)\,\mathrm{d}\tau \tag{3}$$

where, $\varepsilon_2(r, t)$ denotes the normal strain and $p(r, t)$ denotes the normal stress in the binder. η_ε is the coefficient of viscosity in the normal compression mode.

The shear stress–strain relationship in the binder is given by

$$\dot{\gamma}_2(r,\theta,t) = \frac{1}{G_2}\dot{q}(r,\theta,t)+\frac{1}{\eta_\gamma}q(r,\theta,\tau) \tag{4}$$

or in its integral representation, the relation reads

$$\gamma_2(r,\theta,t) = \frac{q(r,\theta,t)}{G_2}+\frac{1}{\eta_\gamma}\int_0^t q(r,\theta,\tau)\,\mathrm{d}\tau \tag{5}$$

where $\gamma_2(r, \theta, t)$ denotes the shear strain and $q(r, \theta, t)$ denotes the shear stress in the binder. η_γ is the coefficient of viscosity for the shearing mode.

For the Voigt modes, the normal stress–strain relationship of the binder is given by

$$p(r,t) = E_2\varepsilon_2(r,t)+\eta_\varepsilon\dot{\varepsilon}_2(r,t) \tag{6}$$

or in its integral representation:

$$\varepsilon_2(r,t) = \int_0^t \frac{p(r,\tau)}{\eta_\varepsilon}\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau. \tag{7}$$

The shear stress–strain relationship for the Voigt binder is given by

$$q(r,\theta,t) = G_2\gamma_2(r,\theta,t)+\eta_\gamma\dot{\gamma}_2(r,\theta,t) \tag{8}$$

or in its integral representation:

$$\gamma_2(r,\theta,t) = \int_0^t \frac{q(r,\theta,\tau)}{\eta_\gamma} e^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,d\tau. \tag{9}$$

We are interested in deriving the normal and tangential compliance relationships of this two elastic particle system with a visco-elastic binder. The equations that govern the rate-dependent relationships between forces and the relative movement of two particles are formulated separately for the normal mode and the tangential mode.

Normal compliance

The relative normal approach $\delta_z(t)$ for the two particles is separated into two components, both are time dependent: the normal displacement at the binder–particle interface relative to the particle's centroid, $w_1(r,t)$; and the normal displacement at the binder–particle interface (i.e., at $z = h(r)$ relative to the $z = 0$ plane), $w_2(r,t)$, given by

$$\delta_z(t) = w_1(r,t)+w_2(r,t). \tag{10}$$

We approximate the normal strain to be uniform in the z direction across the thin layer of binder. Thus the normal displacement $w_2(r,t)$ can be expressed as follows:

$$w_2(r,t) = h(r)\frac{p(r,t)}{E_2}+\frac{h(r)}{\eta_\varepsilon}\int_0^t p(r,\tau)\,d\tau \tag{11}$$

where $p(r,\tau)$ is the interfacial normal pressure between the particle and the binder.

We assume that the characteristic dimension of the particle is much larger than that of the particle-binder contact area. It is therefore justifiable to pursue the analysis of $w_1(r,t)$ based on a half-space premise. Following the well-known Boussinesq equation, $w_1(r,t)$ can be related to $p(r,t)$ by:

$$w_1(r,t) = \frac{(1-\nu_1^2)}{\pi E_1}\int_0^a p(\rho,t)\frac{I(\rho,r)\rho}{\sqrt{\rho^2+r^2}}\,d\rho \tag{12}$$

where $I(\rho,r)$ is defined as

$$I(\rho,r) = I(k) = \int_0^{2\pi}\frac{d\theta}{\sqrt{1-k\cos\theta}}$$

$$k = \frac{2r\rho}{r^2+\rho^2}. \tag{13}$$

By summing up the two components $w_1(r,t)$ and $w_2(r,t)$, the relative normal approach $\delta_z(t)$ for the two contact bodies is

$$\delta_z(t) = h(r)\frac{p(r,t)}{E_2}+\frac{h(r)}{\eta_\varepsilon}\int_0^t p(r,\tau)\,d\tau+\frac{1-\nu_1^2}{\pi E_1}\int_0^a p(\rho,t)\frac{I(\rho,r)\rho}{\sqrt{\rho^2+r^2}}\,d\rho. \tag{14}$$

Integration of the interfacial pressure function, $p(r,t)$, over the contact area gives the resultant normal contact force $P_z(t)$

$$P_z(t) = 2\pi \int_0^a p(r, t) r \, \mathrm{d}r. \tag{15}$$

Equations (14) and (15) govern the magnitude and distribution of interfacial pressure. The compliance relationship is a time-dependent function that relates the relative normal approach $\delta_z(t)$ and the contact force $P_z(t)$.

Tangential compliance

Similarly, the relative tangential approach in the x-direction $\delta_x(t)$ for the two particles is also separated into two components: the tangential displacement at the binder–particle interface relative to the particle's centroid, $u_1(r, \theta, t)$; and the tangential displacement at the binder–particle interface (i.e., at $z = h(r)$) relative to the $z = 0$ plane, $u_2(r, \theta, t)$, given by

$$\delta_x(t) = u_1(r, \theta, t) + u_2(r, \theta, t). \tag{16}$$

Considering a uniform shear strain in the thin layer of binder, the following relation can then be derived:

$$u_2(r, \theta, t) = h(r) \frac{q(r, \theta, t)}{G_2} + \frac{h(r)}{\eta_x} \int_0^t q(r, \theta, \tau) \, \mathrm{d}\tau \tag{17}$$

where G_2 is the shear modulus of binder, $q(r, \theta, t)$ is the interfacial tangential pressure between the particle and the binder.

We again use the assumption that the particle dimension is much larger than that of the contact area, thus the relationship between $u_1(r, \theta, t)$ and $q(r, \theta, t)$ are known based on the half-space premise (Johnson, 1985).

$$u_1(r, \theta, t) = \frac{1}{2\pi G_1} \int_0^{2\pi} \int_0^a q(\rho, \phi, t) F(r, \rho, \theta, \phi, \nu_1) \rho \, \mathrm{d}\rho \, \mathrm{d}\phi \tag{18}$$

where

$$F(r, \rho, \theta, \phi, \nu_1) = \left\{ \frac{1 - \nu_1}{\xi} + \nu_1 \frac{(r \cos\theta - \rho \cos\phi)^2}{\xi^3} \right\}$$

$$\xi^2 = (r \cos\theta - \rho \cos\phi)^2 + (r \sin\theta - \rho \sin\phi)^2, \tag{19}$$

and the values of G_1 and ν_1 are, respectively, the shear modulus and Poisson's ratio of the particle.

From the summation of $u_1(r, \theta, t)$ and $u_2(r, \theta, t)$, governing integral equation corresponding to the Maxwell binder becomes:

$$\delta_x(t) = h(r) \frac{q(r, \theta, t)}{G_2} + \frac{h(r)}{\eta_x} \int_0^t q(r, \theta, \tau) \, \mathrm{d}\tau + \frac{1}{2\pi G_1} \int_0^{2\pi} \int_0^a q(\rho, \phi, t) F(r, \rho, \theta, \phi, \nu_1) \rho \, \mathrm{d}\rho \, \mathrm{d}\phi. \tag{20}$$

Integration of the interfacial pressure function, $q(r, \theta, t)$, over the contact area gives the resultant tangential force $P_x(t)$

$$P_x(t) = \int_0^{2\pi}\int_0^a q(r,\theta,t) r \, dr \, d\theta. \quad (21)$$

The time-dependent tangential compliance relationship between the relative tangential approach $\delta_x(t)$ and the contact for $P_x(t)$ is described by eqns (20) and (21) through the interfacial pressure function, $q(r, \theta, t)$.

Both the governing eqns (14) for normal mode and (20) for tangential mode are Fredholm integral equations of the second kind with kernels containing a logarithmic singularity. The rate-dependent compliance relationship can be determined by simultaneously solving eqns (14) and (15), or eqns (20) and (21). Although the solutions can be pursued using a numerical discretization technique similar to that in Zhu *et al.* (1995), we focus are attentions in this paper on deriving closed-form analytical solutions for the rate-dependent compliance relationship.

The solutions derived in the following sections are illustrated in detail for Maxwell type visco-elastic binder. The methods of deriving solutions for Voigt type binder are similar to that for Maxwell type binder. Therefore, we will only list the final solutions for the Voigt type binder.

SOLUTIONS FOR TWO EXTREME CASES

The exact solutions of the interfacial pressures $p(r, t)$ in eqn (14) and $q(r, \theta, t)$ in eqn (20) are known for two extreme cases, namely, (1) rigid particle case (i.e., $E_1 \to \infty$ and $G_1 \to \infty$ while E_2 and G_2 are finite), and (2) rigid binder case (i.e., E_1 and G_2 are finite while $E_2 \to \infty$ and $G_1 \to \infty$).

Rigid particle case

In the rigid particle case, the relative movement of the two contact bodies is contributed only from the time-dependent deformation of visco-elastic binder. Thus

$$\delta_z(t) = h(r)\frac{p(r,t)}{E_2} + \frac{h(r)}{\eta_\varepsilon}\int_0^t p(r,\tau)\,d\tau \quad (22)$$

$$\delta_x(t) = h(r)\frac{q(r,\theta,t)}{G_2} + \frac{h(r)}{\eta_\gamma}\int_0^t q(r,\theta,\tau)\,d\tau. \quad (23)$$

The corresponding normal and tangential interfacial pressures denoted as $p_1(r, t)$, and $q_1(r, t)$ is given by

$$p_1(r,t) = \frac{P_z(t)}{\pi a^2}\frac{h_0}{h(r)X} \quad (24)$$

$$q_1(r,t) = \frac{P_x(t)h_0}{\pi a^2 h(r)X} \quad (25)$$

where

$$X = \frac{\ln(1+d)}{d} \quad (26)$$

and d is the shape parameter defined in eqn (1).

Thus the time-dependent normal compliance relationship between the contact force $P_z(t)$ and the relative approach $\delta_z(t)$ becomes

$$\delta_x(t) = C_{1z}P_z + C_{1z}\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,d\tau \tag{27}$$

$$C_{1z} = \frac{h_0}{\pi a_2 E_2 X} \tag{28}$$

and the corresponding tangential compliance is:

$$\delta_x(t) = C_{1x}P_x(t) + C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,d\tau \tag{29}$$

$$C_{1x} = \frac{h_0}{\pi a^2 G_2 X}. \tag{30}$$

Rigid binder case

In the rigid binder case, the deformation is contributed only from the particle. The normal interfacial pressure denoted as $p_2(r, t)$ and the tangential pressure denoted as $q_2(r, t)$ corresponding to the rigid punch problem are known to be

$$p_2(r, t) = \frac{P_z(t)}{2\pi a}(a^2 - r^2)^{-1/2} \tag{31}$$

$$q_2(r, t) = \frac{P_x(t)}{2\pi a}(a^2 - r^2)^{-1/2}. \tag{32}$$

For this case, the normal and tangential compliances are:

$$\delta_z = C_{2z}P_z(t); \quad C_{2z} = \frac{1 - \nu_1^2}{2aE_1} \tag{33}$$

$$\delta_x(t) = C_{2x}P_x(t); \quad C_{2x} = \frac{2 - \nu_1}{8aG_1}. \tag{34}$$

UPPER BOUND SOLUTION

Although the explicit solutions of eqn (14) and eqn (20) are easily derived for the two extreme conditions, it is difficult to obtain the analytical solutions of the two equations under general conditions. Therefore, we use the approach given in Zhu *et al.* (1995) to seek the approximate solutions which represent the upper and lower bounds. Based on the bounds, we then seek for the best estimated solutions.

Similar to the approach of finding an upper bound solution in elastic case, we first alter the governing eqn (14) by multiplying $r/h(r)$ and then integrating the equation over the range $0 \leqslant r \leqslant a$, which yields:

$$\delta_z(t) = C_{1z}P_z(t) + C_{1z}\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,d\tau + C_{2z}\frac{4h_0}{\pi X}\int_0^a f(\rho)p(\rho, t)\rho\,d\rho \tag{35}$$

where C_{1z} and C_{2z} are the compliances of the two extreme cases given previously in eqns (28) and (33), and the kernel function

$$f(\rho) = \int_0^a \frac{I(\rho, r) r \, \mathrm{d}r}{h(r)\sqrt{(r^2+\rho^2)}}. \tag{36}$$

Similarly, for the tangential compliance, we multiply $r/h(r)$ to eqn (20), then integrating the equation with respect to the variables (r, θ) over the range $0 < r < a$, $0 < \theta < 2\pi$, we obtain the following expression:

$$\delta_x(t) = C_{1x} P_x(t) + C_{1x} \frac{G_2}{\eta_?} \int_0^t P_x(\tau)\,\mathrm{d}\tau$$
$$+ C_{2x} \frac{4h_0}{a\pi^2 X(2-v_1)} \int_0^{2\pi} \int_0^a f_x(\rho, \phi, v_1) q(\rho, \phi, t) \rho \,\mathrm{d}\rho \,\mathrm{d}\phi \tag{37}$$

where C_{1x} and C_{2x} are the compliances of the two extreme cases given previously in eqns (30) and (34), and

$$f_x(\rho, \phi, v_1) = \int_0^{2\pi} \int_0^a F(\rho, r, \theta, \phi, v_1) r \,\mathrm{d}r \,\mathrm{d}\theta. \tag{38}$$

The original set of governing eqns (14) and (20) are now expressed in a different form as shown in eqns (35) and (37) which are in terms of the compliances of the two extreme conditions. Due to the aded complication of time dependency, it is difficult to verify the monotonic properties of the two functions $p(\rho, t)\rho$ and $q(\rho, \phi, t)\rho$. Therefore, we do not pursue the upper and lower bound solutions based on the principles of *Chebyshev's inequality for integrals*.

Instead, we use an alternative method which requires only the monotonic property of the kernel function $f(\rho)$. Using the property that $f(\rho)$ decreases monotonically in $0 \leqslant \rho \leqslant a$ (see Zhu *et al.*, 1995), the integral in eqn (35) is upper bound when $f(\rho)$ is replaced by $f(0)$. The expression of $f(0)$ is given in Appendix B of Zhu *et al.* (1995). Thus the following inequality is derived:

$$\int_0^a f(\rho) p(\rho, t) \rho \,\mathrm{d}\rho \leqslant f(0) \int_0^a p(\rho, t) \rho \,\mathrm{d}\rho = \frac{a P_z(t)}{h_0 \sqrt{d}} tg^{-1}\sqrt{d}. \tag{39}$$

Substituting eqn (39) into (25), the upper bound solution for the time-dependent normal compliance is derived as

$$\delta_z(t) \leqslant (C_{1z} + b_1 C_{2z}) P_z(t) + C_{1z} \frac{E_2}{\eta_E} \int_0^t P_z(\tau)\,\mathrm{d}\tau \tag{40}$$

where

$$b_1 = \frac{4}{\pi X} \frac{tg^{-1}\sqrt{d}}{\sqrt{d}} > 1. \tag{41}$$

Note that $b_1 = 4/\pi = 1.273$ when $d = 0$ and $b_1 = 1/\ln 2 = 1.443$ when $d = 1$.

For tangential compliance, it is easily seen that

$$f_x(\rho, \phi, v_1) \leqslant f_x(\rho, \phi, 0) = 2\pi f(\rho) \tag{42}$$

where $f(\rho)$ is the kernel function defined in eqn (36). We can use both eqns (42) and (37) to derive the following inequality:

$$\delta_x(t) \leqslant C_{1x}P_x(t)+C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,\mathrm{d}\tau+C_{2x}\frac{8h_0}{a\pi X(2-v_1)}\int_0^{2\pi}\int_0^a f(\rho)q(\rho,\phi,t)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi. \tag{43}$$

Similarly, the replacement of $f(\rho)$ by $f(0)$ in the integral of eqn (43) leads to the upper bound solution for tangential compliance:

$$\delta_x(t) \leqslant \left(C_{1x}+\frac{2b_1}{2-v_1}C_{2x}\right)P_x(t)+C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,\mathrm{d}\tau. \tag{44}$$

where b_1 is defined in eqn (41).

LOWER BOUND SOLUTION

For the lower bound analysis, we convert the original set of governing equations by multiplying $p_2(r,t)r$ by eqn (14), integrating the equation over the range $0 \leqslant r \leqslant a$, and by multiplying $q_2(r,t)r$ by eqn (20), integrating the equation over the range $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\phi$, thus:

$$\delta_z(t) = \frac{2\pi}{\eta_\varepsilon P_z(t)}\int_0^t\int_0^a p(r,\tau)rp_2(r,\tau)h(r)\,\mathrm{d}r\,\mathrm{d}\tau+\frac{2\pi}{E_2P_z(t)}\int_0^a p(r,t)rp_2(r,t)h(r)\,\mathrm{d}r+C_{2z}P_z(t) \tag{45}$$

$$\delta_x(t) = \frac{1}{\eta_\gamma P_x(t)}\int_0^t\int_0^{2\pi}\int_0^a q(r,\theta,\tau)rq_2(r,\tau)h(r)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\tau + \frac{1}{G_2P_x(t)}\int_0^{2\pi}\int_0^a q(r,\theta,t)rq_2(r,t)h(r)\,\mathrm{d}r\,\mathrm{d}\theta+C_{2x}P_x(t). \tag{46}$$

From eqns (1), (31) and (32), it is easily seen that $p_2(r,t)$ and $q_2(r,t)h(r)$ increase monotonically in the range $0 \leqslant r \leqslant a$. Thus the replacement of $p_2(r,t)h(r)$ by $p_2(0,t)h(0)$ in the integral of eqn (45) results in the following inequality:

$$\int_0^a p(r,\tau)rp_2(r,\tau)h(r)\,\mathrm{d}r \geqslant p_2(0,\tau)\int_0^a p(r,\tau)r\,\mathrm{d}r = \frac{P_z(\tau)}{2\pi a^2}\frac{P_z(\tau)}{2\pi} \tag{47}$$

and the lower bound solution for the normal compliance is derived as

$$\delta_z(t) \geqslant (b_2C_{1z}+C_{2z})P_z(t)+C_{1z}b_2\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,\mathrm{d}\tau \tag{48}$$

where

$$b_2 = \frac{X}{2} < 1. \tag{49}$$

The function X is defined in eqn (26). Note that $b_2 = 0.5$ when $d = 0$ and $b_2 = 0.347$ when $d = 1$.

Similarly, the replacement of $q_2(r, t)h(r)$ by $q_2(0, t)h(0)$ in eqn (46) results in the lower bound solution for tangential compliance:

$$\delta_x(t) \geq (b_2 C_{1x} + C_{2x}) P_x(t) + C_{1x} b_2 \frac{G_2}{\eta_\gamma} \int_0^t P_x(\tau)\, d\tau. \tag{50}$$

BEST ESTIMATE BASED ON PHYSICAL APPROXIMATIONS

In this section, we seek for the best estimated compliance relationship. Instead of using the previous approach of simplifying the integral governing equations into inequalities, we now approach the problem by selecting a suitable form of pressure function that can be substituted directly into the governing equations. Thus the best estimated solution can be obtained. Two estimates are conducted: the first estimate is based on the set of governing eqns (35) and (37); the second estimate is based on the set of governing eqns (45) and (46).

In the first estimate, we select the interfacial pressure $p_2(\rho, t)$ given in eqn (31) for the rigid punch problem as the substituting pressure function for $p(\rho, t)$ in eqn (35). It can be seen that, when C_{1z} is negligible (i.e., rigid binder case), this substitution yields the exact expression of a rigid punch solution. On the other hand, when C_{1z} becomes dominant and C_{2z} is negligible (i.e., the rigid particle case), the contribution of the integral is trivial to the solution of eqn (35), thus the form of pressure function makes little difference. Therefore, substituting the function $p(\rho, t)$ with $p_2(\rho, t)$ is a physically consistent choice, and it leads to the following simple compliance relationship:

$$\delta_z(t) = (C_{1z} + C_{2z}) P_z(t) + C_{1z} \frac{E_2}{\eta_E} \int_0^t P_z(\tau)\, d\tau \tag{51}$$

and its rate-dependent form:

$$\dot{\delta}_z(t) = (C_{1z} + C_{2z}) \dot{P}_z(t) + C_{1z} \frac{E_2}{\eta_E} P_z(t) \tag{52}$$

where the symbol ($\cdot$) denotes the derivative with respect to t.

When eqn (45) serves as the starting point of the second estimate, employing the same argument in deriving the first estimate, we select the interfacial pressure function $p_1(r, t)$ given in eqn (32) for the rigid particle case as the substituting pressure function for $p(r, t)$ in eqn (45). When C_{2z} is negligible (rigid particle case), the substitution yields an exact solution. When C_{2z} becomes dominant (rigid binder case), the contribution of the integral is trivial in eqn (45) and the form of pressure function makes little difference to the compliance. Thus the second best estimated solution is obtained by substituting the $p(\rho, t)$ in eqn (45) with $p_1(\rho, t)$, and it yields, surprisingly, the identical relationship to the one in first estimate (i.e., eqn (51)).

Similarly, for tangential compliance, we select the rigid binder pressure $q_2(\rho, t)$ to substitute the unknown pressure distribution $q(\rho, \phi, t)$ in the integral of eqn (37) and the rigid particle pressure $q_1(\rho, t)$ to substitute the unknown pressure distribution $q(r, \theta, t)$ in the integral of eqn (46). Both processes lead to the identical result:

$$\delta_x(t) = (C_{1x} + C_{2x}) P_x(t) + C_{1x} \frac{G_2}{\eta_\gamma} \int_0^t P_x(\tau)\, d\tau \tag{53}$$

and its rate-dependent form:

$$\dot{\delta}_x(t) = (C_{1x}+C_{2x})\dot{P}_x(t) + C_{1x}\frac{G_2}{\eta_\gamma}P_x(t). \tag{54}$$

The best estimated compliance relationships (eqns (51) and (53)) satisfy the two extreme cases: (1), rigid particle case ($E_1 \to \infty$ and E_2 finite); and (2), rigid binder case (E_1 finite and $E_2 \to \infty$). In addition, the best estimated compliance falls in between the upper and lower bounds.

VOIGT MODEL

Accordingly, for the case of Voigt visco-elastic binder, the governing equation of the interfacial pressure and the relative approach of two bodies becomes

$$\delta_z(t) = \frac{h(r)}{\eta_\varepsilon}\int_0^t p(r,\tau)\,\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau + \frac{1-v_1^2}{\pi E_1}\int_0^a p(\rho,t)\frac{I(\rho,r)\rho}{\sqrt{\rho^2+r^2}}\,\mathrm{d}\rho \tag{55}$$

for the normal compliance, and

$$\delta_x(t) = \frac{h(r)}{\eta_\gamma}\int_0^t q(r,\theta,\tau)\,\mathrm{e}^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,\mathrm{d}\tau + \frac{1}{2\pi G_1}\int_0^{2\pi}\int_0^a F(r,\rho,\theta,\phi,v_1)q(\rho,\phi,t)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \tag{56}$$

for the tangential compliance.

The derivation process of compliance relationships for Voigt binder is very similar to that for Maxwell binder. For simplicity, we omit the derivation and list only the final results which include: the upper bound solutions, the lower bound solutions, and the best estimated solutions. The upper bound solutions are

$$\delta_z(t) \leqslant b_1 C_{2z}P_z(t) + C_{1z}\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau \tag{57}$$

$$\delta_x(t) \leqslant \frac{2b_1}{2-v_1}C_{2x}P_x(t) + C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,\mathrm{e}^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,\mathrm{d}\tau. \tag{58}$$

The lower bound solutions are

$$\delta_z(t) \geqslant C_{2z}P_z(t) + b_2 C_{1z}\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau \tag{59}$$

$$\delta_x(t) \geqslant C_{2x}P_x(t) + C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,\mathrm{e}^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,\mathrm{d}\tau. \tag{60}$$

The best estimate solutions are

$$\delta_z(t) = C_{2z}P_z(t)+C_{1z}\frac{E_2}{\eta_\varepsilon}\int_0^t P_z(\tau)\,\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau \tag{61}$$

$$\delta_x(t) = C_{2x}P_x(t)+C_{1x}\frac{G_2}{\eta_\gamma}\int_0^t P_x(\tau)\,\mathrm{e}^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,\mathrm{d}\tau \tag{62}$$

and its rate-dependent versions are

$$\dot{\delta}_z(t) = C_{1z}\frac{E_2}{\eta_\varepsilon}P_z(t)+C_{2z}\dot{P}_z(t)-C_{1z}\left(\frac{E_2}{\eta_\varepsilon}\right)^2\int_0^t P_z(t)\,\mathrm{e}^{-\frac{E_2}{\eta_\varepsilon}(t-\tau)}\,\mathrm{d}\tau \tag{63}$$

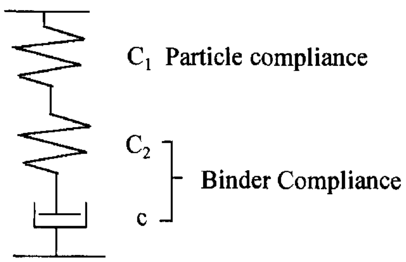
$$\dot{\delta}_x(t) = C_{1x}\frac{G_2}{\eta_\gamma}P_x(t)+C_{2x}\dot{P}_x(t)-C_{1x}\left(\frac{G_2}{\eta_\gamma}\right)^2\int_0^t P_x(t)\,\mathrm{e}^{-\frac{G_2}{\eta_\gamma}(t-\tau)}\,\mathrm{d}\tau \tag{64}$$

or equivalent,

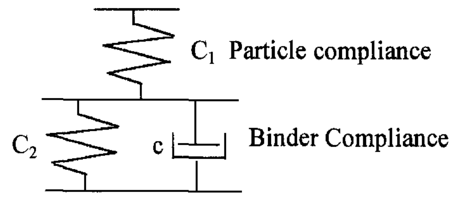
$$\dot{\delta}_z(t)+\frac{E_2}{\eta_\varepsilon}\delta_z(t) = C_{2z}\dot{P}_z(t)+(C_{1z}+C_{2z})\frac{E_2}{\eta_\varepsilon}P_z(t) \tag{65}$$

$$\dot{\delta}_x(t)+\frac{G_2}{\eta_\gamma}\delta_x(t) = C_{2x}\dot{P}_x(t)+(C_{1x}+C_{2x})\frac{G_2}{\eta_\gamma}P_x(t). \tag{66}$$

The solutions presented here suggest that the rate-dependent compliance of a system comprised two elastic particles with a visco-elastic binder can be simulated by an equivalent spring-dashpot system. We observe that the compliances can be schematically shown in Fig. 1a and Fig. 1b for two different types of visco-elastic binder. The compliance of the elastic particle is represented by a spring. The compliance of a visco-elastic binder of Maxwell type is represented by a serial connection of a spring and a dashpot while the binder of Voigt type is represented by a parallel connection of a spring and a dashpot. When the binder is elastic, the system is reduced to a serial connection of two springs as



(a) Maxwell Binder



(b) Voigt Binder

Fig. 1. Two types of equivalent spring-dashpot system.

described in Fig. 3 in Zhu *et al.* (1995). Note that the dashpot coefficient c has the unit of force-time/length while the viscosity η of the binder has the unit of force-time/length square.

The compliance relationships between $\delta_z(t)$ vs $P_z(t)$ and between $\delta_x(t)$ vs $P_x(t)$ for the equivalent spring-dashpot system of Maxwell model (i.e., Fig. 1a) are given by

$$\dot{\delta}_z(t) = (C_{1z}+C_{2z})\dot{P}_z(t)+\frac{P_z(t)}{c_z} \tag{67}$$

$$\dot{\delta}_x(t) = (C_{1x}+C_{2x})\dot{P}_x(t)+\frac{P_x(t)}{c_x}. \tag{68}$$

For the equivalent spring-dashpot system of Voigt model (i.e., Fig. 1b), the compliance relationships are given by

$$\dot{\delta}_z(t)+\frac{\delta_z(t)}{C_{1z}c_z} = C_{2z}\dot{P}_z(t)+\frac{C_{1z}+C_{2z}}{C_{1z}c_z}P_z(t) \tag{69}$$

$$\dot{\delta}_x(t)+\frac{\delta_x(t)}{C_{1x}c_x} = C_{2x}\dot{P}_x(t)+\frac{C_{1x}+C_{2x}}{C_{1x}c_x}P_x(t) \tag{70}$$

where C_{1z}, C_{2z}, C_{1x} and C_{2x} are the spring constants, and c_z and c_x are the dashpot coefficents for the equivalent system.

For the Maxwell model, we compare the pair of eqns (67) and (68) for the spring-dashpot system with the pair of analytical solutions (eqns (52) and (54)) for the particle–binder system. The comparison yields that C_{1z}, C_{2z}, C_{1x} and C_{2x} are the spring compliances for the equivalent system. The dashpot coefficients c_z and c_x for the equivalent system are:

$$c_z = \frac{\eta_E}{E_2C_{1z}} \tag{71}$$

$$c_x = \frac{\eta_\gamma}{G_2C_{1x}}. \tag{72}$$

For the Voigt model, we compare the pair of eqns (69) and (70) for the spring-dashpot system with the pair of eqns (65) and (66) for the particle–binder system. The comparison yields the identical spring compliances and equivalent dashpot coefficients as those obtained from the Maxwell model. This property of model independence indicates that it is a plausible approach to use the spring and dashpot elements for the simulation of an assembly of particles.

CONCLUSIONS

We derive the rate-dependent compliance relationship for a system of two elastic particles bonded by a thin layer of visco-elastic binder system. Since the governing equations for the system are also Fredholm integral equations of the second kind, the previous approach for elastic particle–binder system is adopted to obtain the solutions of compliance relationship. Due to the difficulties of proving the monotonic properties for the two time-dependent functions $p(\rho, t)\rho$ and $q(\rho, \phi, t)\rho$, the principles of *Chebyshev's inequality* are not adopted in this paper for the problem with visco-elastic binder. Instead, we derive the upper and lower bound solutions based on the monotonic property of the kernel function $f(\rho)$. This alternative method simplifies the governing equations thus yields simple closed-form solutions for the upper and lower bound rate-dependent compliances. Similar to the results for elastic binder system, the derived upper bound solutions (eqns (40) and (44)) involve a constant b_1 greater than one while the derived lower bound solutions (eqns (48) and (50)) involve a constant b_2 less than one. The best estimated solutions (eqns (52) and (54)) correspond to $b_1 = b_2 = 1$.

The derived results show that the rate-dependent compliance relationship for a particle–binder system is equivalent to that of a spring-dashpot system. Expressions for the spring constant representing the particle compliance are given in eqns (28) and (3). The compliance of the visco-elastic Maxwell or Voigt binder is represented by a serial or a parallel connection of a spring and a dashpot. Expressions for the spring constant representing the elastic compliance of the binder are given in eqns (33) and (34). Expressions for the dashpot coefficient representing the viscous compliance of the binder are given in eqns (71) and (72). This concept of an equivalent spring-dashpot system is potentially useful to the analysis of assemblies with a large number of particles bonded by visco-elastic binders.

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